Find the Fourier series of the function:

(i) $f(x) = \begin{cases} 0 & \text{if } -\pi < x < 0 \\ 1 & \text{if } 0 < x < \pi \end{cases}$

→ we can use the Euler formulas;

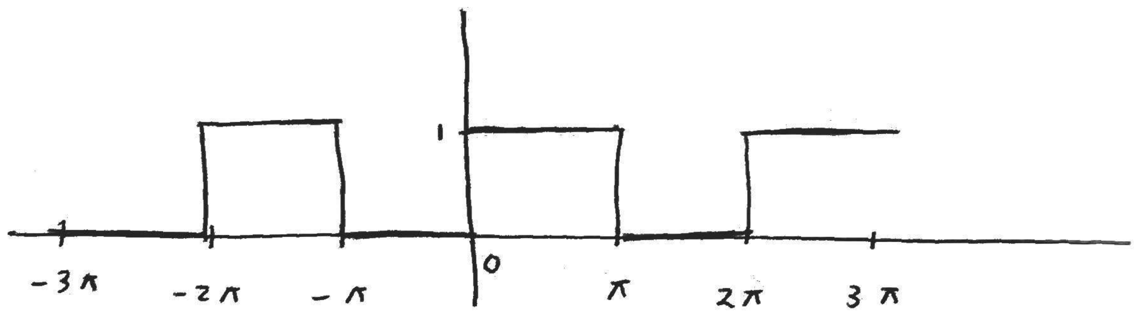
$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right)$$

where;

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx$$



$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{0} 0\, dx + \frac{1}{2\pi} \int_{0}^{\pi} 1\, dx$$

$$a_0 = \frac{1}{2\pi} \int_{0}^{\pi} dx = \frac{1}{2\pi} x \Big|_0^{\pi} = \frac{1}{2\pi} [\pi - 0]$$

$$= \boxed{\frac{1}{2}}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{0} (0) \cos(nx)\, dx + \frac{1}{\pi} \int_{0}^{\pi} (1) \cos(nx)\, dx$$

$$a_n = \frac{1}{\pi} \int_{0}^{\pi} \cos(nx)\, dx = \frac{1}{\pi} \left[\frac{1}{n} \sin(nx) \right]_0^{\pi}$$

$$a_n = \frac{1}{\pi} \left[\frac{1}{n} \sin(n\pi) - \frac{1}{n} \sin(0) \right]$$

$$a_n = \frac{1}{n\pi} \left[0 - 0 \right] = \boxed{0}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{0} (0) \sin(nx)\, dx + \frac{1}{\pi} \int_{0}^{\pi} (1) \sin(nx)\, dx$$

$$b_n = \frac{1}{\pi} \int_{0}^{\pi} \sin(nx)\, dx = \frac{1}{\pi} \left[-\frac{1}{n} \cos(nx) \right]_0^{\pi}$$

$$b_n = \frac{1}{n\pi} \left[-\cos(n\pi) - (-\cos(0)) \right]$$

$$b_n = \frac{1}{n\pi} \left[-\cos(n\pi) + 1 \right]$$

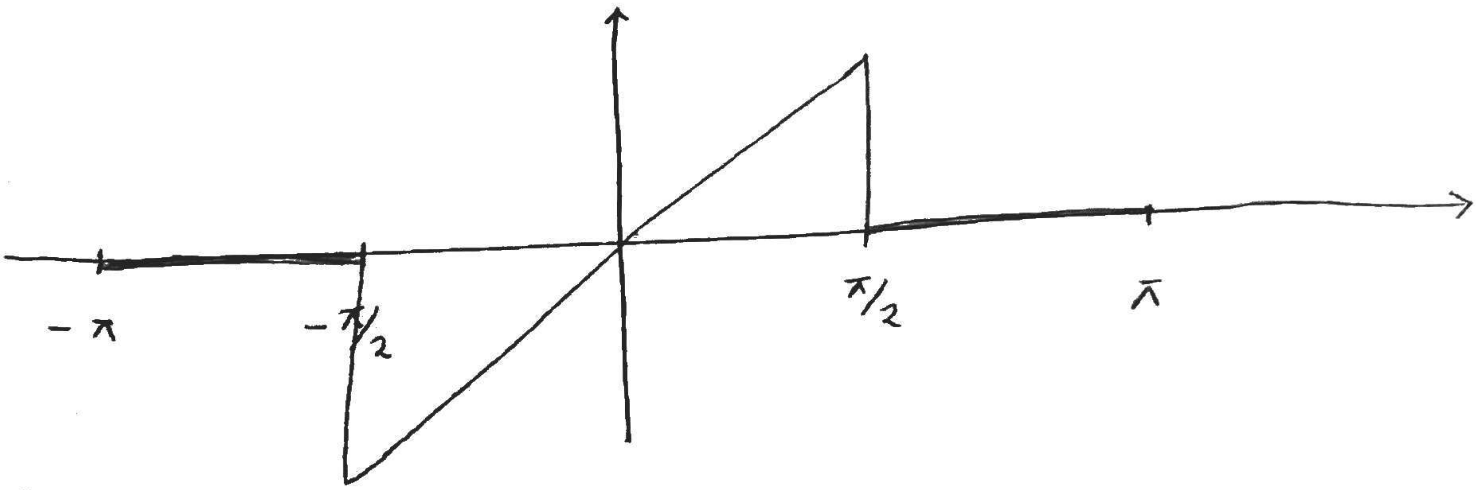
$$\cos(n\pi) = \begin{cases} 1 & \text{for } n \text{ even} \\ -1 & \text{for } n \text{ odd} \end{cases} = (-1)^n$$

$$b_n = \frac{1}{n\pi} \left[1 - (-1)^n \right] = \begin{cases} \frac{1}{n\pi} \left[1 - (-1) \right] = \frac{2}{n\pi} & \text{for } n \text{ odd} \\ \frac{1}{n\pi} \left[1 - 1 \right] = 0 & \text{for } n \text{ even} \end{cases}$$

$$\Rightarrow \boxed{f(x) = \frac{1}{2} + \sum_{\substack{n=1 \\ \text{for} \\ n \text{ odd}}}^{\infty} \frac{2}{n\pi} \sin(nx)}$$

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\left[\sin(x) + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \dots\right]$$

(ii) $f(x) = \begin{cases} x & \text{if } |x| < \pi/2 \\ 0 & \text{if } \pi/2 < |x| < \pi \end{cases}$



$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

$$\rightarrow a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$a_0 = \frac{1}{2\pi}\left[\int_{-\pi}^{-\pi/2} 0\, dx + \int_{-\pi/2}^{\pi/2} x\, dx + \int_{\pi/2}^{\pi} 0\, dx\right]$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} x\, dx = \frac{1}{2\pi}\left[\frac{x^2}{2}\right]_{-\pi/2}^{\pi/2}$$

$$a_0 = \frac{1}{\cancel{4}\pi}\left[\left(\frac{\pi}{2}\right)^2 - \left(-\frac{\pi}{2}\right)^2\right] = \boxed{0}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$\rightarrow$ from $-\pi$ to $-\pi/2$ and $\pi/2$ to π; $f(x) = 0$

$\Rightarrow$ don't need to consider these

$$a_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} x\cos(nx)\,dx$$

integrate by parts; $\int u\,dv = uv - \int v\,du$

$\rightarrow$ $u = x$; $du = dx$

$\rightarrow$ $dv = \cos(nx)\,dx$

$$v = \int \cos(nx)\,dx = \frac{1}{n}\sin(nx)$$

$$\Rightarrow \int x\cos(nx)\,dx = \frac{x}{n}\sin(nx) - \int \frac{1}{n}\sin(nx)\,dx$$

$$= \frac{1}{\pi}\left[\frac{x}{n}\sin(nx) + \frac{1}{n^2}\cos(nx) \Big|_{-\pi/2}^{\pi/2} \right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2n}\sin\left(\frac{n\pi}{2}\right) + \frac{1}{n^2}\cos\left(\frac{n\pi}{2}\right) + \frac{\pi}{2n}\sin\left(-\frac{n\pi}{2}\right) - \frac{1}{n^2}\cos\left(-\frac{n\pi}{2}\right) \right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2n}\sin\left(\frac{n\pi}{2}\right) + \frac{1}{n^2}\cos\left(\frac{n\pi}{2}\right) - \frac{\pi}{2n}\sin\left(\frac{n\pi}{2}\right) - \frac{1}{n^2}\cos(n\pi/2) \right] = \boxed{0}$$

$$b_n = \int_{-\pi/2}^{\pi/2} x \sin(nx)\, dx$$

integrate by parts; $\int u\,dv = uv - \int v\,du$

$$u = x$$
$$du = dx$$
$$dv = \sin(nx)\,dx$$
$$v = \int \sin(nx)\,dx = -\frac{1}{n}\cos(nx)$$

$$\rightarrow b_n = \frac{1}{\pi}\left[-\frac{x}{n}\cos(nx) + \int \frac{1}{n}\cos(nx)\,dx\right]_{-\pi/2}^{\pi/2}$$

$$b_n = \frac{1}{\pi}\left[-\frac{x}{n}\cos(nx) + \frac{1}{n^2}\sin(nx)\right]_{-\pi/2}^{\pi/2}$$

$$b_n = \frac{1}{\cancel{\pi}}\left[-\frac{\cancel{\pi}}{2n}\cos\left(\frac{n\pi}{2}\right) + \frac{1}{\pi n^2}\sin\left(\frac{n\pi}{2}\right)\right]$$

$$- \frac{1}{\cancel{\pi}}\left[-\left(-\frac{\cancel{\pi}}{2n}\cos\left(\frac{-\pi n}{2}\right) + \frac{1}{\pi n^2}\sin\left(\frac{n\pi}{2}\right)\right)\right]$$

$$b_n = -\frac{1}{2n}\cos\left(\frac{n\pi}{2}\right) + \frac{1}{\pi n^2}\sin\left(\frac{n\pi}{2}\right)$$

$$- \frac{1}{2n}\cos\left(\frac{n\pi}{2}\right) + \frac{1}{\pi n^2}\sin\left(\frac{n\pi}{2}\right)$$

$$b_n = -\frac{1}{n}\cos\left(\frac{n\pi}{2}\right) + \frac{2}{\pi n^2}\sin\left(\frac{n\pi}{2}\right)$$

$$\Rightarrow \boxed{f(x) = \sum_{n=1}^{\infty}\left(\frac{2}{\pi n^2}\sin\left(\frac{n\pi}{2}\right) - \frac{1}{n}\cos\left(\frac{n\pi}{2}\right)\right)\sin(nx)}$$